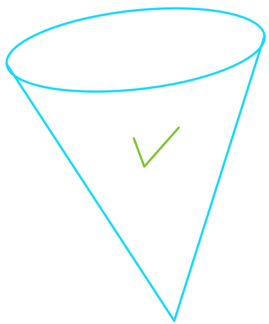
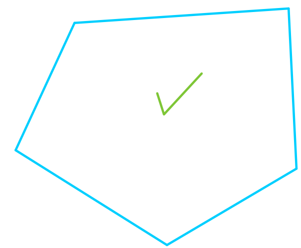
Intro to

Convexity Zine

— by Rachel Shi





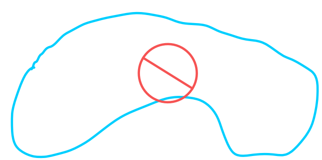
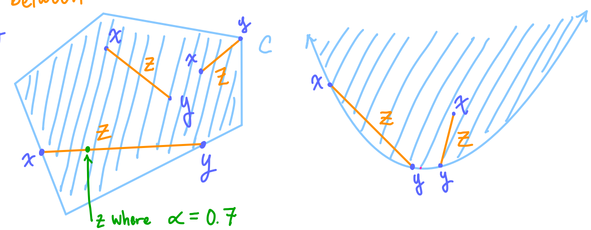


Table Of Contents

What is a convex cone?

a set C is convex if for any two elements $x, y \overset{\text{in}}{\in} C$,

the element: $\underbrace{z = \alpha x + (1-\alpha) y}_{\text{notice that all such points } z \text{ form a chord between points } x \text{ and } y} \overset{\text{in}}{\in} C \quad \overset{\text{for all}}{\forall} \; \underbrace{\alpha \in [0,1]}_{\alpha \text{ between 0 and 1 (including 0 and 1)}}$



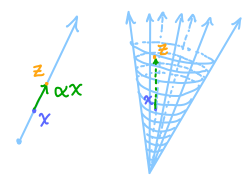
examples of convex sets

In other words, C is convex if any line segment between two elements in C is also in C.

---

a set C is a cone if $x \overset{\text{in}}{\in} C$,

then $z = \alpha x \overset{\text{in}}{\in} C \quad \overset{\text{for all}}{\forall} \; \underbrace{\alpha \geq 0}_{\alpha \text{ that is not negative}}$



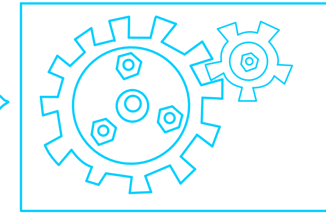
examples of cones

---

combining these two ideas, a <u>convex cone</u> is a set C where if $x, y \overset{\text{in}}{\in} C$

then $z = \alpha_x x + \alpha_y y \overset{\text{in}}{\in} C \quad \overset{\text{for all}}{\forall} \; \underbrace{\alpha_x, \alpha_y \geq 0}_{\alpha_x \text{ and } \alpha_y \text{ are both greater than } 0}$

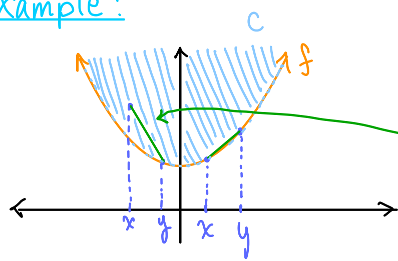
Convex Functions



given a convex set, C, a function on C, $f: C \to \mathbb{R}$, is convex if $\forall\, x, y \in C$ and $\forall\, \alpha \in [0,1]$:

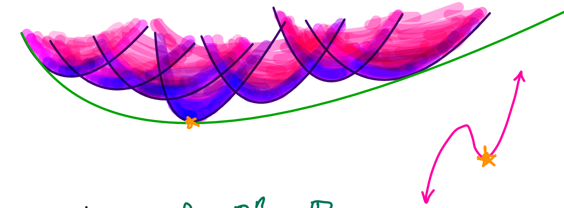
$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$$

example:



intuitively, this means that for any 2 points in C, the line segment between them lies ABOVE the graph of convex function f.

General Optimization



the general form of a mathematical optimization problem is:

problem

- minimize/maximize $f_0(x)$ ← an objective function! where $f_0 : \mathbb{R}^n \to \mathbb{R}$
- subject to $f_i(x) \leq b_i$ ← constraint functions! where $f_i : \mathbb{R}^n \to \mathbb{R}$
  for $i = 1, \ldots, m$
- x (a vector) is our optimization variable!

solution

we usually call an optimal value of x, x^* ↳ called "x-star"
which is the solution to our problem

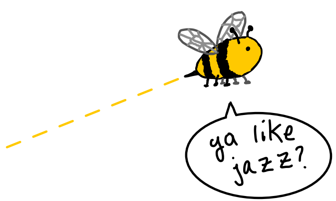
Known as LP! → linear programming: a special case of the general optimization problem

Constraints

where all constraint functions are linear

↳ this means that each of the functions $f_0, \ldots, f_m$ satisfy:

$$\boxed{f_i(\alpha x + \beta y) = \alpha f_i(x) + \beta f_i(y)}$$

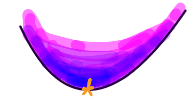


linear programs are usually written in the form:

$$
\begin{aligned}
\text{maximize} \quad & c^T x \\
\text{subject to} \quad & Ax \leq b \\
\text{and} \quad & x \geq 0
\end{aligned}
$$

→ where A is a matrix and b is a vector

Convex Optimization



Like linear programming, convex optimization is a special case of the general optimization problem,

constraints where all constraint functions are convex

↳ this means that each of the functions $f_0, \dots, f_m$ satisfy:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

Comparing this to linear programing where

constraint functions are linear

↳ this means that each of the functions $f_0, \dots, f_m$ satisfy:

$$\boxed{f_i(\alpha x + \beta y) = \alpha f_i(x) + \beta f_i(y)}$$

we see that linear programming is a special case of convex optimization